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JORDAN ALGEBRAS WITH MINIMUM CONDITION(1)

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Abstract. Let J be a Jordan algebra with minimum condition on quadratic ideals over a field of characteristic not 2. We construct a maximal nil ideal R of J such that J/R is a direct sum of a finite number of ideals each of which is a simple Jordan algebra. R must have finite dimension if it is nilpotent and this is shown to be the case whenever J has "enough" connected primitive orthogonal idempotents.

Introduction. The first four sections of this paper deal with the construction of a maximal nil ideal for a Jordan algebra J with minimum condition on quadratic ideals over a field F of characteristic not 2. We call this ideal, R, the radical of J and show that J/R is semisimple in the sense of Jacobson [2].

We construct R by "building up" through the Peirce decomposition of J relative to a finite collection of primitive orthogonal idempotents. In order to facilitate computations, much of the construction is carried out under the assumption that J has an identity element. However, once R is constructed we show that this assumption is not really necessary.

In the last section we deal with questions of the nilpotence of R and use the Coordinatization Theorem as our basic tool. We show that R is nilpotent if J has enough connected primitive orthogonal idempotents. Also if R is nilpotent, then it must be finite dimensional.

1. **Preliminaries.** In this paper an algebra will be an algebra over a field F of characteristic not 2, which is not necessarily associative or of finite dimension. J will always denote a commutative Jordan algebra. In order to make this paper self-contained we now recall some definitions and results from [2]. If $x \in J$, U_x denotes the linear operator $2R_x^2 - R_{x^2}$ on J, and if $x, y, z \in J$ then $\{xyz\}$ will denote the trilinear product $xy\cdot z + yz\cdot x - xz\cdot y$.

$$U_x^n = U_{x^n}, \qquad U_yU_x = U_xU_{yU_x}, \qquad zU_{x+y} = zU_x + zU_y + 2\{xzy\}.$$

An element $0 \neq b \in J$ is called an absolute zero divisor of J if and only if $JU_b = 0$.

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A subspace Q of J is a quadratic ideal if and only if $JU_x \subset Q$ for all $x \in Q$.

If e is an idempotent of J, then the quadratic ideal JU_e is the Peirce space $J_1(e)$. Also any quadratic ideal (absolute zero divisor) of $J_1(e)$ or $J_0(e)$ is also a quadratic ideal (absolute zero divisor) of J.

If J has an identity element 1, then an element $x \in J$ is invertible if and only if U_x^n is invertible for any positive integer n. Also if $1 \in J$, then every quadratic ideal of J is also a subalgebra of J.

Finally, any homomorph of a quadratic ideal in J is a quadratic ideal of the homomorph of J.

LEMMA 1.1. *Let A be a nonzero quadratic ideal of J. If there exists an element $a \in A$ such that $A = AU_a$, then A contains an idempotent.*

Proof. Since $A = AU_a$ there is $c \in A$, such that $a = cU_a$. So $U_a = U_aU_cU_a$. If $z \in A$ we can find $w \in A$ with $z = wU_a = wU_aU_cU_a = zU_cU_a$, hence U_cU_a is the identity map on A. In particular $c = cU_cU_a = c^3U_a$. But then $z = zU_cU_a = zU_{c^3U_a}U_a = zU_aU_c^3U_a^2 = zU_aU_c$. Therefore $U_aU_c = U_cU_a$ on A.

If $f = a^2U_c$, then $zU_f = zU_{a^2U_c} = zU_cU_a^2U_a = z$. So U_f is the identity map on A. Hence $f = fU_f = f^3$. So $g = f^2 = (a^2U_c)^2 = c^2U_a^2U_c$ is a nonzero idempotent of A.

We will use a special case of the following result in §3.

LEMMA 1.2. *Suppose J has an identity element 1 and that $b = b_1 + b_{12} + b_0$ is the decomposition of an absolute zero divisor b of J relative to an idempotent e. Then b_1 and b_0 are absolute zero divisors of J.*

Proof. Since $b_0 \in J_1(1-e)$ all we need show is that $b_1 \in J_1(e)$ is an absolute zero divisor of J. $0 = 1U_b = b^2 = b_1^2 + [b_{12}^2]_1 + 2(b_1 + b_0)b_{12} + b_0^2 + [b_{12}^2]_0$ so $0 = b_1^2 + [b_{12}^2]_1 = (b_1 + b_0)b_{12} = b_0^2 + [b_{12}^2]_0$.

If $x_1 \in J_1(e)$ then

$$\begin{aligned} 0 &= x_1U_b = 2x_1b \cdot b - x_1b^2 = 2x_1b \cdot b = 2[x_1(b_1 + b_{12} + b_0)](b_1 + b_{12} + b_0) \\ &= 2x_1b_1 \cdot b_1 + 2[x_1b_{12} \cdot b_{12}]_1 + 2[x_1b_{12} \cdot b_1 + x_1b_1 \cdot b_{12} + x_1b_{12} \cdot x_0]_{12} + 2[x_1b_{12} \cdot b_{12}]_0. \end{aligned}$$

But then $x_1b_1 \cdot b_1 = -[x_1b_{12} \cdot b_{12}]_1$ for all $x_1 \in J_1(e)$. In particular $eb_1 \cdot b_1 = b_1^2 = -[eb_{12} \cdot b_{12}]_1 = -\frac{1}{2}[b_{12}^2]_1$. So we have that $b_1^2 = 0$. Using the Jordan identity we get

$$\begin{aligned} 0 &= (e, b_{12}, x_1b_{12}) + (x_1, b_{12}, eb_{12}) + (b_{12}, b_{12}, x_1e) \\ &= \tfrac{1}{2}x_1b_{12} \cdot b_{12} - [x_1b_{12} \cdot b_{12}]_1 + \tfrac{1}{2}x_1b_{12} \cdot b_{12} - \tfrac{1}{2}x_1b_{12}^2 + x_1b_{12}^2 - x_1b_{12} \cdot b_{12} \\ &= -[x_1b_{12} \cdot b_{12}]_1. \end{aligned}$$

Therefore $x_1U_{b_1} = 2x_1b_1 \cdot b_1 - x_1b_1^2 = 0$ and b_1 is an absolute zero divisor of $J_1(e)$ hence also of J.

Henceforth, all Jordan algebras under consideration in this paper are assumed to have minimum condition on quadratic ideals. Such Jordan algebras equivalently have descending chain condition on quadratic ideals.

THEOREM 1.3. *J is nil or J contains an idempotent.*

Proof. Let $x \in J$ and consider the descending chain of quadratic ideals $J \supset JU_x \supset JU_{x^2} \supset \cdots$. By assumption there must be a positive integer n so that $JU_{x^n} = JU_{x^{n+k}}$ for $k=1, 2, \ldots$. Hence $JU_{x^n}=0$ or JU_{x^n} is a quadratic ideal satisfying the hypothesis of Lemma 1.1. So either x is nilpotent or JU_{x^n}, and hence J, will contain an idempotent.

LEMMA 1.4. *If e is an idempotent of J then $J_1(e)$ contains a primitive idempotent of J.*

Proof. Clearly $J_1(e)$ has minimum condition on quadratic ideals since every quadratic ideal of $J_1(e)$ is also a quadratic ideal of J. Let $Q=JU_f=J_1(f)$, $f^2=f \in J_1(e)$, be a minimal quadratic ideal of $J_1(e)$ determined by idempotents of $J_1(e)$. If f is not primitive, then Q contains an idempotent $f_1 \neq f$ and clearly $JU_{f_1} \subsetneqq JU_f = Q$, which is contrary to the choice of Q.

THEOREM 1.5. *If J is not nil, then J contains a principal idempotent u where $u=\sum_{i=1}^n e_i$ where the e_i are primitive orthogonal idempotents of J.*

Proof. Since J is not nil we know that J contains an idempotent e; which by Lemma 1.3 can be assumed to be primitive. Suppose that $e_1, e_2, \ldots, e_k$ are primitive orthogonal idempotents of J. If $J_0(e_1+\cdots+e_k)$ contains an idempotent it contains a primitive one, e_{k+1}, hence $e_1, e_2, \ldots, e_k, e_{k+1}$ are primitive orthogonal idempotents of J. One easily sees, however, that

$$J_0(e_1) \supset J_0(e_1+e_2) \supset \cdots \supset J_0(e_1+e_2+\cdots+e_{k+1})$$

is a properly descending chain of quadratic ideals. Therefore this chain must come to an end after a finite number of steps, say n. So we let $u=\sum_{i=1}^n e_i$ and clearly u is principal since $J_0(u)$ contains no idempotents.

COROLLARY 1.6. *If J has an identity element 1 then $1=\sum_{i=1}^n e_i$, where the e_i are primitive orthogonal idempotents.*

If J does not have an identity element then we may imbed J in $J'=F1 \oplus J$ in the usual way. We now prove

THEOREM 1.7. *$J'=F1 \oplus J$ also has minimum condition on quadratic ideals.*

Proof. Let $Q_1 \supset Q_2 \supset \cdots \supset Q_n \supset$ be a properly descending chain of quadratic ideals of J'. We consider the descending chain of quadratic ideals

$$Q_1 \cap J \supset Q_2 \cap J \supset \cdots \supset Q_n \cap J \supset \cdots$$

of J. Without loss of generality we may assume that $Q_1 \cap J = Q_2 \cap J$ and that $Q_1, Q_2 \not\subset J$. Since $Q_1 \supsetneqq Q_2$, we know that there is an element $a \in J$ so that $1+a \in Q_1 \backslash Q_2$ and since $Q_2 \not\subset J$ there is an element $b \in J$ so that $1+b \in Q_2$. But then $(1+a)(1+b) = 1+a+b+ab \in Q_1$, hence $b+ab$, $a+ab \in Q_1 \cap J = Q_2 \cap J \supset Q_2$. But this implies that $1+a \in Q_2$, a contradiction. So with the possible exception of Q_1, our chain is in J, hence it has finite length and therefore J' has minimum condition on quadratic ideals.

2. J **contains only one idempotent.** Throughout this section we assume that $1 \in J$ and that 1 is the only idempotent of J. We show that every element in J is either invertible or nilpotent and that the nilpotent elements form an ideal.

THEOREM 2.1. *Every element of J is either invertible or nilpotent.*

Proof. By the proof of Theorem 1.3 we see that if $x \in J$ is not nilpotent then JU_x must contain an idempotent which must be 1. Hence x is invertible.

THEOREM 2.2. *J has nilpotent elements if and only if J has absolute zero divisors.*

Proof. If J has absolute zero divisors then clearly J has nilpotent elements.

If J has no absolute zero divisors then by Theorem 2 of [2], J is either division or J contains an idempotent different from the identity. By our hypothesis, J must be division, hence contains no nilpotent elements.

We now let N be the collection of all nilpotent elements of J and we define

$$K = \{x \in J \mid x+N \subset N\}.$$

LEMMA 2.3. *$K \subset N$, K is a subspace of J, and $K \neq \{0\}$ whenever $N \neq \{0\}$.*

Proof. Clearly K is a subspace of J contained in N. By the previous lemma we know that if $N \neq \{0\}$, J contains an absolute zero divisor b. If $n \in N$ we know there is a positive integer $k>1$ so that $n^k=0$ and $n^{k-1} \neq 0$. Now

$$n^{k-1}U_{b+n} = n^{k-1}U_b+n^{k-1}U_n+2\{bn^{k-1}n\} = 2[bn^{k-1}\cdot n-bn\cdot n^{k-1}] = 0$$

since $R_{x^p}R_{x^q}=R_{x^q}R_{x^p}$ in any Jordan algebra. Hence U_{b+n} is not invertible so $b+n$ is not invertible, so by Theorem 2.1, $b+n$ is nilpotent and $b \in K$.

LEMMA 2.4. *If $s \in J$ is invertible then $NU_s \subset N$.*

Proof. Since s is invertible we have that $JU_s=JU_s^{-1}=J$. If $n \in N$ and $J=JU_{nU_s}$ then $J=JU_sU_nU_s=JU_nU_s$. Therefore $J=JU_n$ and n is invertible a contradiction. So $nU_s \in N$.

LEMMA 2.5. *If $s \in J$ is invertible then $KU_s \subset K$.*

Proof. Let $k \in K$ and $n \in N$. Since s is invertible, U_s is invertible and so there exists $m \in N$ such that $mU_s=n$. Therefore $n+kU_s=mU_s+kU_s=(m+k)U_s \subset N$, hence $kU_s \in K$.

LEMMA 2.6. *K is a nil ideal of J.*

Proof. Let $k \in K$ and suppose first that $x \in J$ is invertible. If $1+x$ is also invertible then $kU_{1+x}=k+kU_x+2\{1kx\} \in K$ and so $kx \in K$. If $1+x \in N$ then $1+(1+x)$ is invertible so again $kU_{1+(1+x)}=kU_2+kU_x+2\{2kx\} \in K$. But this also implies that $kx \in K$. Hence if $x \in J$ is invertible $Kx \subset K$.

If $x \in N$, then $1+x$ is invertible, hence $k(1+x)=k+kx \in K$ so again $kx \subset K$. But then $KJ \subset K$ and $K \subset N$, so K is a nil ideal of J.

THEOREM 2.7. *N is an ideal of J.*

Proof. If $N \neq \{0\}$ then $K \neq \{0\}$ is a nil ideal of J. Let M be a maximal nil ideal of J. Since homomorphic images of invertible (nilpotent) elements are invertible (nilpotent) in the homomorph, we see that the identity of J/M is the only idempotent of J/M, and so J/M satisfies the same conditions as J does. If $M \neq N$, then J/M contains nilpotent elements and therefore has a nontrivial nil ideal. But this is a contradiction to the choice of M, and hence $N=M$ is a nil ideal of J.

COROLLARY 2.8. *J/N is a division algebra.*

3. **Peirce decomposition relative to two primitive orthogonal idempotents.** In the Peirce decomposition $J_{11}+J_{12}+J_{22}$ of J relative to two primitive orthogonal idempotents e_1 and e_2, it is clear that both J_{11} and J_{22} have the minimum condition on quadratic ideals and the only idempotents of J_{11} and J_{22} are their respective identity elements e_1 and e_2. So the collection, N_i, of nilpotent elements of J_{ii} is an ideal of J_{ii} for $i=1, 2$. We show in this section that $N_1+(N_1+N_2)J_{12}+N_2$ is a nil ideal of J. The proof is by a series of computational lemmas, so throughout this section we assume that $n_i \in N_i$, all letters or expressions with subscripts 12 and i are assumed to be elements of J_{12} and J_{ii} respectively, $i=1, 2$.

LEMMA 3.1. $J_{ii}U_{y_{12}} \subset J_{jj}$, *$i, j=1, 2$; $i \neq j$.*

Proof. $0=(x_i, y_{12}, y_{12}e_j)+(y_{12}, y_{12}, x_ie_j)+(e_j, y_{12}, y_{12}x_i)=x_iy_{12}\cdot y_{12}-[x_iy_{12}\cdot y_{12}]_j-\frac{1}{2}x_iy_{12}^2$. Hence $x_iU_{y_{12}}=2x_iy_{12}\cdot y_{12}-x_i\cdot y_{12}^2=2[x_iy_{12}\cdot y_{12}]_j \in J_{jj}$.

LEMMA 3.2. $x_{12}^{2k+1}=2x_{12}([x_{12}^2]_i)^k$, *$i=1, 2$; $k=1, 2, \ldots$.*

Proof. $x_{12}\cdot e_ix_{12}^2=x_{12}e_i\cdot x_{12}^2$. Hence $x_{12}[x_{12}^2]_i=\frac{1}{2}x_{12}^3$, and we have $2x_{12}[x_{12}^2]_i=x_{12}^3$. If $x_{12}^{2k-1}=2x_{12}([x_{12}^2]_i^{k-1})$ then $x_{12}^{2k+1}=x_{12}^{2k-1}x_{12}^2=2(x_{12}[x_{12}^2]_i^{k-1})x_{12}^2=2x_{12}([x_{12}^2]_i^k)$. By induction we have our lemma.

COROLLARY 3.3. *Every element in J_{12} is either invertible or nilpotent.*

Proof. This follows easily from the preceding lemma since $x_{12}^2 \in J_{11}+J_{22}$.

LEMMA 3.4. $[(n_iy_{12})^2]_i \in N_i$ *and hence n_iy_{12} is nilpotent for $i=1, 2$.*

Proof.

$$0 = 2(n_i, y_{12}, n_iy_{12})+(y_{12}, y_{12}, n_i^2) = 2(n_iy_{12})^2-2n_i(n_iy_{12}\cdot y_{12})+y_{12}^2n_1^2-n_i^2y_{12}\cdot y_{12}.$$

But N_i is an ideal of J_{ii} and so $[(n_iy_{12})^2]_i \in N_i$ if and only if $[n_1^2y_{12}\cdot y_{12}]_i \in N_i$. But $[n_iy_{12}\cdot y_{12}]_i=\frac{1}{2}n_iy_{12}^2$ for any $n_i \in N_i$ by the proof of Lemma 3.1, and so $[(n_iy_{12})^2]_i \in N_i$.

LEMMA 3.5. $n_iy_{12}\cdot y_{12} \in N_1+N_2$, *$i=1, 2$.*

Proof. First suppose that y_{12} is invertible. If $n_iU_{y_{12}}$ is invertible in J_{jj} then

$$J_{jj} = J_{jj}U_{n_iU_{y_{12}}} = J_{jj}U_{y_{12}}U_{n_i}U_{y_{12}} \subset J_{ii}U_{n_i}U_{y_{12}} \subset N_iU_{y_{12}} \subset J_{jj}$$

and so $J_{jj}=N_iU_{y_{12}}$. Then $J_{jj}U_{y_{12}}=N_iU_{y_{12}}^2=N_iU_{y_{12}^2}\subset N_i$. But then $e_jU_{y_{12}^2}=[y_{12}^2]_i \in N_i$ and thus y_{12} is nilpotent, a contradiction. So $n_iU_{y_{12}}=2[n_iy_{12}\cdot y_{12}]_j \in N_j$ and thus $n_iy_{12}\cdot y_{12} \in N_1+N_2$.

If y_{12} is nilpotent then $(n_iy_{12}+y_{12})^2=(n_iy_{12})^2+2n_iy_{12}\cdot y_{12}+y_{12}^2=(n_iy_{12})^2+2[n_iy_{12}\cdot y_{12}]_i+2[n_iy_{12}\cdot y_{12}]_j+[y_{12}^2]_i+[y_{12}^2]_j$. But $[(n_iy_{12})^2]_i$, $[n_iy_{12}\cdot y_{12}]_i$, $[y_{12}^2]_i$ are all elements of N_i, so $n_iy_{12}+y_{12}$ is nilpotent and $n_iy_{12}\cdot y_{12} \in N_1+N_2$.

LEMMA 3.6. (a) $[n_iy_{12}\cdot z_{12}]_j \in N_j$, $i, j=1, 2$, $i\neq j$;
(b) $[n_iy_{12}\cdot z_{12}]_i+[n_iz_{12}\cdot y_{12}]_i \in N_i$.

Proof. Linearizing the result in Lemma 3.5 we see that $n_iy_{12}\cdot z_{12}+n_iz_{12}\cdot y_{12} \in N_1+N_2$ and clearly (b) holds.

$$\begin{aligned} 0 &= (n_i, y_{12}, y_{12}e_i)+(e_i, y_{12}, n_iz_{12})+(z_{12}, y_{12}, n_ie_i) \\ &= \tfrac{1}{2}(z_{12}, y_{12}, n_i)+(e_i, y_{12}, n_iz_{12}) \\ &= \tfrac{1}{2}n_i\cdot y_{12}z_{12}-\tfrac{1}{2}n_iy_{12}\cdot z_{12}+\tfrac{1}{2}n_iz_{12}\cdot y_{12}-[n_iz_{12}\cdot y_{12}]_i. \end{aligned}$$

Multiplying through by e_j we see that $[n_iy_{12}\cdot z_{12}]_j=[n_iz_{12}\cdot y_{12}]_j$ and hence

$$[n_iy_{12}\cdot z_{12}]_j \in N_j.$$

LEMMA 3.7. $n_iy_{12}\cdot z_{12} \in N_1+N_2$, $i=1, 2$.

Proof. We have already shown that $[n_iy_{12}\cdot z_{12}]_j \in N_j$ so we will now show that $[n_iy_{12}\cdot z_{12}]_i \in N_i$ by looking at $([n_iy_{12}\cdot z_{12}]_i)^2=[(n_iy_{12}\cdot z_{12})^2]_i$.

$$\begin{aligned} 0 &= 2(n_iy_{12}, z_{12}, n_iy_{12}\cdot z_{12})+(z_{12}, z_{12}, (n_iy_{12})^2) \\ &= 2(n_iy_{12}\cdot z_{12})^2-2(n_iy_{12})((n_iy_{12}\cdot z_{12})z_{12})+(z_{12}, z_{12}, (n_iy_{12})^2). \end{aligned}$$

By Lemmas 3.4 and 3.5 $(z_{12}, z_{12}, (n_iy_{12})^2) \in N_1+N_2$, so $[(n_iy_{12}\cdot z_{12})^2]_i \in N_i$ if and only if $[(n_iy_{12})((n_iy_{12}\cdot z_{12})z_{12})]_i \in N_i$. If we now let $w_{12}=n_iy_{12}$,

$$\begin{aligned} 0 &= 2(z_{12}, e_i, w_{12}z_{12})+(w_{12}, e_i, z_{12}^2) \\ &= [w_{12}z_{12}]_jz_{12}-[w_{12}z_{12}]_iz_{12}-\tfrac{1}{2}w_{12}[z_{12}^2]_i+\tfrac{1}{2}w_{12}[z_{12}^2]_j. \end{aligned}$$

And so $[w_{12}z_{12}]_iz_{12}=[w_{12}z_{12}]_jz_{12}-\tfrac{1}{2}w_{12}[z_{12}^2]_i+\tfrac{1}{2}w_{12}[z_{12}^2]_j$. Hence

$$\begin{aligned} [w_{12}(w_{12}z_{12}\cdot z_{12})]_i &= [w_{12}(z_{12}[w_{12}z_{12}]_i+z_{12}[w_{12}z_{12}]_j)]_i \\ &= [w_{12}(z_{12}[w_{12}z_{12}]_j)]_i+\tfrac{1}{2}[w_{12}\cdot w_{12}[z_{12}^2]_j]_i-\tfrac{1}{2}[w_{12}\cdot w_{12}[z_{12}^2]_i]_i. \end{aligned}$$

Now $[w_{12}z_{12}]_j=[n_iy_{12}\cdot z_{12}]_j \in N_j$, therefore by Lemma 3.6(a), $[w_{12}(z_{12}[w_{12}z_{12}]_j)]_i \in N_i$. We now consider $[w_{12}\cdot w_{12}[z_{12}^2]_j]_i$ and $[w_{12}\cdot w_{12}[z_{12}^2]_i]_i$. Clearly

$$w_{12}\cdot w_{12}[z_{12}^2]_j = w_{12}(n_iy_{12}\cdot [z_{12}^2]_j) = w_{12}(n_i\cdot y_{12}[z_{12}^2]_j) = n_iy_{12}\cdot n_ix_{12}$$

with $x_{12}=y_{12}[z_{12}^2]_j$. By Lemma 3.4

$$[n_i(y_{12}+x_{12})]^2 = (n_iy_{12})^2+2n_iy_{12}\cdot n_ix_{12}+(n_ix_{12})^2 \in N_1+N_2$$

and so $[w_{12}\cdot w_{12}[z_{12}^2]_j]_i \in N_i$. Also $[w_{12}\cdot w_{12}[z_{12}^2]_i]_i$ is in N_i since again as in the proof of Lemma 3.1, $[w_{12}\cdot w_{12}[z_{12}^2]_i]_i = \frac{1}{2}w_{12}^2[z_{12}^2]_i$ which is in N_i because $w_{12}^2 \in N_i$. So finally we have proven that $n_i y_{12}\cdot z_{12} = [n_i y_{12}\cdot z_{12}]_i + [n_i y_{12}\cdot z_{12}]_j \in N_1 + N_2$.

We can now easily prove the following:

THEOREM 3.7. *$N_1+(N_1+N_2)J_{12}+N_2$ is a proper nil ideal of J.*

Proof. Clearly $N=N_1+(N_1+N_2)J_{12}+N_2$ is a proper ideal of J by using the well-known multiplication properties of the Peirce subspaces J_{ij} of J, and the fact that N_i is an ideal of J_{ii}.

Using the proof of Theorem 1.3 we see that N is either nil or N contains an idempotent $e=n_1+x_{12}+n_2$. But then since $e^2=e$ we have $\frac{1}{2}x_{12}=x_{12}(n_1+n_2)$. Letting $n=n_1+n_2$ we see that n and hence R_n are nilpotent where $xR_n=xn$. But clearly then $x_{12}=0$ which contradicts e being an idempotent. Hence N is a nil ideal of J.

We will need the next two lemmas in §4.

LEMMA 3.8. *Suppose that J_{11} and J_{22} are division algebras. Then $b_{12}\in J_{12}$ is an absolute zero divisor of J if and only if $b_{12}J_{12}=0$.*

Proof. $0=2(y_{12}, b_{12}, y_{12}b_{12})+(b_{12}, b_{12}, y_{12}^2)=2(b_{12}y_{12})^2-2(y_{12}b_{12}\cdot b_{12})y_{12}+b_{12}^2y_{12}^2-y_{12}^2b_{12}\cdot b_{12}=2(b_{12}y_{12})^2$. But $b_{12}y_{12}\in J_{11}+J_{22}$ and so $b_{12}y_{12}=0$, i.e. $b_{12}J_{12}=0$.

If $c_{12}J_{12}=0$, then $xU_{c_{12}}=2xc_{12}\cdot c_{12}-xc_{12}^2=2(x_1+x_{12}+x_2)c_{12}\cdot c_{12}=2x_1c_{12}\cdot c_{12}+2x_{12}c_{12}\cdot c_{12}+x_2c_{12}\cdot c_{12}=0$. Since $x_ic_{12}\in J_{12}$, $i=1, 2$. So c_{12} is an absolute zero divisor of J.

LEMMA 3.9. *If both J_{11} and J_{22} are division algebras, then the collection B_{12} of absolute zero divisors of J is an ideal of J.*

Proof. By the previous lemma it is clear that B_{12} is a subspace of J_{12}. If $x_i\in J_{ii}$, $i=1, 2$ and $b_{12}\in B_{12}$, then $0=(x_i, b_{12}, y_{12}e_j)+(y_{12}, b_{12}, e_jx_i)+(e_j, b_{12}, y_{12}x_i)=\frac{1}{2}x_ib_{12}\cdot y_{12}$. Hence $x_ib_{12}\in B_{12}$ and $J_{ii}B_{12}\subset B_{12}$ which implies that B_{12} is an ideal of J.

4. **The Peirce decomposition in general.** If J is not nil, we recall that J must contain a principal idempotent u which is the sum of finite number of primitive orthogonal idempotents $e_1, \dots, e_n$. We show that the $N_i=\{x\in J_{ii} \mid x \text{ is nilpotent}\}$ generate an ideal N so that the Peirce one spaces in J/N are division algebras. If J/N has absolute zero divisors then we show there is a nil ideal R of J with $N\subset R$, such that J/R has no absolute zero divisors, and J/R is semisimple.

LEMMA 4.1. *If N_i is the collection of nilpotent elements of J_{ii} in the Peirce decomposition of J then*

(1) $N_iJ_{ij}\cdot J_{ij}\subset N_i+N_j$, $N_iJ_{ij}\cdot J_{ii}\subset N_iJ_{ij}$, $N_iJ_{ij}\cdot J_{jj}\subset N_iJ_{ij}$;

(2) $N_iJ_{ij}\cdot J_{jk}\subset N_iJ_{ik}$, $i, j, k\neq$;

(3) $N_iJ_{ij}\cdot J_{ik}\subset N_iJ_{ik}\cdot J_{ij}$, $i, j, k\neq$;

(4) $(N_iJ_{ij}\cdot J_{ik})J_{ik}\subset N_iJ_{ij}$, $i, j, k\neq$;

(5) $(N_iJ_{ij}\cdot J_{ik})J_{jk}\subset N_{jj}+N_{kk},\ i, j, k\neq;$
(6) $(N_iJ_{ij}\cdot J_{ik})J_{kk}\subset N_iJ_{ij}\cdot J_{ik},\ i, j, k\neq;$
(7) $(N_iJ_{ij}\cdot J_{ik})J_{kl}\subset N_iJ_{ij}\cdot J_{il},\ i, j, k, l\neq.$

Using this lemma we easily prove

THEOREM 4.2. *If* $J=\sum_{i\leq j=1}^{n} J_{ij}$ *is the Peirce decomposition of J relative to the primitive orthogonal idempotents* $e_1, \ldots, e_n$, *where* $1=\sum_{i=1}^{n} e_i$ *and* $N_i=\{x\in J_{ii} \mid x$ *is nilpotent*$\}$, *then* $N=\sum N_i+\sum N_iJ_{ij}+\sum N_iJ_{ij}\cdot J_{ik}$ *is a proper ideal of J.*

If we let φ_i be the restriction of the natural homomorphism $\varphi: J\to J/N$ to J_{ii} we set that $\varphi_i: J_{ii}\to J_{ii}/N_i$ and so $\varphi_i(J_{ii})$ is a division algebra. J/N may have absolute zero divisors. However, they are easily handled by the following lemma and theorem.

LEMMA 4.3. *If* $J=\sum J_{ij}$ *is the Peirce decomposition of J relative to the orthogonal idempotents* $e_1, \ldots, e_n$, *where* $1=\sum e_i$ *is the identity element of J, and if* J_{ii} *is a division algebra for* $i=1, 2, \ldots, n$ *then if b is an absolute zero divisor of J,* $b=\sum_{i<j} b_{ij}$ *where each* $b_{ij}\in J_{ij}$ *is an absolute zero divisor of J.*

Proof. Let $b=b_1+b_{12}+b_0$ be the decomposition of b relative to e_i. Then by Lemma 1.2, b_1 is an absolute zero divisor of J_{ii} and so $b_1=0$. Hence $b=\sum_{i<j} b_{ij}$. If $f=e_i+e_j$, and if $b=c_1+c_{12}+c_0$ is the decomposition of b relative to f we know that c_1 is an absolute zero divisor of J, $c_1\in J_1(f)=J_{ii}+J_{ij}+J_{jj}$. But the only component of b in $J_{ii}+J_{ij}+J_{jj}$ is b_{ij} so $b_{ij}=c_1$ is an absolute zero divisor of J.

THEOREM 4.4. *If* J_{ii} *is a division algebra for* $i=1, \ldots, n$ *in the Peirce decomposition of J relative to the orthogonal idempotents* $e_1, \ldots, e_n$ *with* $1=\sum e_i$, *then*
(1) $A_0=\sum_{i<j} B_{ij}$ *is an ideal of J where* $B_{ij}=\{x\in J_{ij} \mid xJ_{ij}=0\}$,
(2) J/A_0 *has no absolute zero divisors.*

Proof. We will show that $b_{ij}y_{jk}\in B_{ik}$ where $b_{ij}\in B_{ij}$ and $y_{jk}\in J_{jk}$. By Lemma 3.8, b_{ij} is an absolute zero divisor of J, so $0=2(y_{jk}, b_{ij}, y_{jk}b_{ij})+(b_{ij}, b_{ij}, y_{jk}^2)=2(b_{ij}y_{jk})^2$, also $0=(y_{jk}, b_{ij}, x_{ik}e_i)+(e_i, b_{ij}, y_{jk}x_{ik})+(x_{ik}, b_{ij}, y_{jk}e_i)=\frac{1}{2}(b_{ij}y_{jk}\cdot x_{ik}-b_{ij}x_{ik}\cdot y_{jk})$ so $b_{ij}y_{jk}\cdot x_{ik}=b_{ij}x_{ik}\cdot y_{jk}$. By comparing components we see that $b_{ij}y_{jk}\cdot x_{ik}=[b_{ij}y_{jk}\cdot x_{ik}]_k$. If we let $z_{ik}=b_{ij}y_{jk}$, then $z_{ik}^2=0$ and $z_{ik}x_{ik}=[z_{ik}x_{ik}]_k\in J_{kk}$.

$$0 = 2(z_{ik}, e_k, x_{ik}z_{ik})+(x_{ik}, e_k, z_{ik}^2) = z_{ik}\cdot x_{ik}z_{ik}-[2z_{ik}x_{ik}]_kz_{ik} = -z_{ik}x_{ik}\cdot z_{ik}.$$

So $x_{ik}U_{z_{ik}}=0$. If $x_{ii}\in J_{ii}$, then $x_{ii}U_{z_{ik}}=w_{kk}\in J_{kk}$. But $w_{kk}^2=(x_{ii}U_{z_{ik}})^2=z_{ik}^2U_{x_{ii}}U_{z_{ik}}=0$. But J_{kk} is division, hence $w_{kk}=0$. So $J_{ii}U_{z_{ik}}=0$. In the same way $J_{kk}U_{z_{ik}}=0$, and so $z_{ik}=b_{ij}y_{jk}$ is an absolute zero divisor of $J_{ii}+J_{ik}+J_{kk}$. Hence $B_{ij}J_{jk}\subset B_{ik}$, and $A_0=\sum_{i<j} B_{ij}$ is a proper ideal of J.

Let α_{ij} be the restriction of the natural map $J\to J/A_0$ to $J_1(e_i+e_j)=J_{ii}+J_{ij}+J_{jj}$. So $\alpha_{ij}: J_1(e_i+e_j)\to J_1(e_i+e_j)/B_{ij}$. Clearly $\alpha_{ij}(J_{ii})$ and $\alpha_{ij}(J_{jj})$ are division algebras so if $\alpha_{ij}(J_1(e_i+e_j))$ contains an absolute zero divisor $c_{ij}+B_{ij}$, we must have that $(c_{ij}+B_{ij})(J_{ij}+B_{ij})=c_{ij}J_{ij}\subset B_{ij}$. But then $c_{ij}J_{ij}=0$ and $c_{ij}\in B_{ij}$ and $\alpha_{ij}(J_1(e_i+e_j))$ cannot contain any absolute zero divisors. So by Lemma 4.3 J/A_0 contains no absolute zero divisors.

COROLLARY 4.5. *The ideal A_0 in Theorem* 4.3 *is finite dimensional.*

Proof. $A_0=\sum_{i<j} B_{ij}$ and every element of B_{ij} is an absolute zero divisor of J. Hence every subspace of B_{ij} is a quadratic ideal of J, and so B_{ij} has finite dimension $i, j=1, 2, \ldots, n$, $i<j$. Therefore A_0 has finite dimension.

THEOREM 4.6. *Let J have identity element* 1. *Then there exists an ideal R of J so that*

(1) *R is a maximal nil ideal of J,*

(2) *J/R has no absolute zero divisors,*

(3) *J/R is the direct sum of a finite number of ideals which are simple Jordan algebras.*

Proof. Let S be any ideal of J such that J/S has no absolute zero divisors. Then the ideal N of Theorem 4.2 must be contained in S, since if not, J/S would contain absolute zero divisors.

J/N clearly satisfies the hypothesis of Theorem 4.4, so if $\bar{J}=J/N$ contains any absolute zero divisors it has an ideal $\bar{A}_0$ such that $\bar{J}/\bar{A}_0$ contains no absolute zero divisors. Let R be the complete inverse image A_0+N of $\bar{A}_0$ under the mapping $J\rightarrow J/N$. Clearly $R\subset S$. Now J/R satisfies axioms (i), (ii), and (iii) of [2] and hence is a direct sum of ideals which are simple Jordan algebras also satisfying axioms (i), (ii), and (iii). If $R\neq S$ then S must contain an idempotent. If S contains no idempotents then $S=R$. Hence R is a nil ideal (again by the proof of Theorem 1.3) which is clearly maximal.

DEFINITION. The ideal R is called the radical of J.

We now will consider the situation in which J does not contain an identity element.

THEOREM 4.7. *If J is not nil, then there exists an ideal R of J such that*

(1) *R is a maximal nil ideal,*

(2) *J/R has no absolute zero divisors,*

(3) *J/R is the direct sum of a finite number of ideals which are simple Jordan algebras,*

(4) *J/R has an identity element.*

Proof. By Theorem 1.5 we know that J contains a principal idempotent $u=\sum_{i=1}^{n} e_i$ where the e_i are primitive orthogonal idempotents of J. So $J=J_1(u)+J_{1/2}(u)+J_0(u)$ and $J_0(u)$ is nil. Let $J'=F1\oplus J$ be the algebra obtained by adjoining an identity element 1 to J. If $g=1-u$, then we know (see [1]): $gu=0$, $J_1'(u)=J_0'(g)=J_1(u)$, $J_{1/2}'(u)=J_{1/2}(u)$; $J_0'(u)=J_1'(g)=J_0(u)+Fg$, and clearly g is primitive. If R' is the radical of J', then by the construction of R' we know that $J_0(u)\subset R'$.

Now $J'=\sum_{i\leq j\leq 1}^{n} J_{ij}+\sum_{i=1}^{n} J_{i0}+J_{00}+Fg$ where

$$J_{ii} = \{x \mid xe_i = x\}, \qquad J_{ij} = J_{1/2}(e_i)\cap J_{1/2}(e_j), \qquad i,j = 1, 2, \ldots, n, \quad i \leqq j;$$

$$J_{i0} = J_{1/2}(e_i)\cap J_{1/2}(u) = J_{1/2}(e_i)\cap J_{1/2}(g), \qquad J_{00} = J_0(u).$$

Note that $J_{00}+Fg=J_1(g)$.

Since J'/R' contains no absolute zero divisors we see that the image of $J_{ii}+J_{i0}+J_1(g)$ is a Jordan algebra with identity element being the sum of two primitive orthogonal idempotents whose one spaces are division. Since every element in J_{i0} is clearly nilpotent, we see that the image of J_{i0} is 0. So if we let $R=J \cap R'$, we see the J/R contains no absolute zero divisors, and that J/R has an identity element (the image of u). Hence our theorem.

COROLLARY 4.8. $\mathrm{Rad}\, J' = \mathrm{Rad}\,(F1 \oplus J) = \mathrm{Rad}\, J$.

5. **Properties of the radical.** If A is an algebra over a field of characteristic not 2, then we may define an algebra A^+ having the same vector space as A but with multiplication "$\circ$" defined by $x \circ y = \frac{1}{2}(xy+yx)$. If A has an involution then $H(A)$, the collection of symmetric elements of A, forms a subalgebra of A^+. No confusion should arise if we denote this subalgebra also by the symbol $H(A)$.

We use Jacobson's Coordinatization Theorem and some results on isotopy for Jordan algebras as tools in this section. See [3].

LEMMA 5.1. *Let D be an algebra with identity element* 1 *over F and let D have an involution "$^-$" so that $H(D)$ is in the nucleus of D. Also let L be a left ideal of D. Then*

$$Q = \{|a_{ij}| \mid a_{11} \in H(D),\ a_{12} \in L,\ a_{21} = \bar{a}_{12},\ b_{22} \in H(D) \cap L,\ a_{ij} = 0 \textit{ otherwise}\}$$

is a quadratic ideal of the Jordan algebra $H(D_n)$, $n>1$, where D is alternative if $n=2$, 3, and associative if $n \geqq 4$.

Proof. It is well known [3] that $H(D_n)$ is a Jordan algebra when D is alternative if $n=2$, 3 or D is associative if $n \geqq 4$. Let $J=H(D_n)$, and let E_{ij} be the matrix units of D_n. $E_{11}, E_{22}, \ldots, E_{nn}$ are orthogonal idempotents of J such that $I=\sum_{i=1}^n E_{ii}$. Clearly Q is isomorphic to the subalgebra

$$\left\{\begin{vmatrix} a & x \\ \bar{x} & b \end{vmatrix} \mid a \in H(D),\ x \in L,\ b \in H(D) \cap L\right\}$$

of $H(D_2)$. We identify Q with this subalgebra and let

$$A = \begin{vmatrix} a & x \\ \bar{x} & b \end{vmatrix} \in Q \quad \text{and} \quad B = \begin{vmatrix} c & y \\ \bar{y} & d \end{vmatrix} \in H(D_2).$$

One easily sees by computation that $BU_A = ABA \in H(D_2)$. Also $(ABA)_{12} = acx + x\bar{y}x + ayb + xdb \in L$ and $(ABA)_{22} = \bar{x}cx + b\bar{y}x + \bar{x}yb + bdb \in H(D) \cap L$. Hence Q is a quadratic ideal of $H(D_2)$, hence of $J_1(E_{11}+E_{22})$, and so finally of $H(D_n)$.

THEOREM 5.2. *Let $1=\sum_{i=1}^n e_i$ in J where the e_i are primitive orthogonal idempotents. Suppose that $n \geqq 3$ and the e_i are connected. Then $R=\mathrm{Rad}\, J$ is nilpotent.*

Proof. Since $n \geqq 3$ and the e_i are connected, the Coordinatization Theorem [3] tells us that there exists an algebra D with an identity element 1 and an involution "$^-$" such that D is associative if $n \geqq 4$, and is alternative with its selfadjoint elements in the nucleus if $n \geqq 3$, so that J is isomorphic to $H(D_n, \gamma)$.

First of all we assume that $\gamma=1$, i.e. $H(D_n,\gamma)=H(D_n)$. Let $L_1\supset L_2\supset\cdots\supset L_n\supset\cdots$ be a descending chain of left ideals of D. Corresponding to each L_k we form a quadratic ideal:

$$Q_k = \{|a_{ij}| \mid a_{11}\in H(D),\ a_{12}\in L_k,\ a_{21} = \bar{a}_{12},\ a_{22}\in L_k\cap H(D),\ a_{ij} = 0 \text{ otherwise}\}$$

of $H(D_n)$. So we have $Q_1\supset Q_2\supset\cdots\supset Q_n$ is a descending chain of quadratic ideals of $H(D_n)$. Since J has minimum condition on quadratic ideals, $H(D_n)$ does, hence D has minimum condition on left ideals.

If D is associative, then the Jacobson radical, Rad D, is nilpotent. Hence Rad $(D_n)=(\text{Rad } D)_n$ is nilpotent.

If D is alternative, then in [8] K. A. Ževlakov has shown that the Smiley radical [7], Rad D, is nilpotent if D is of characteristic not 2 or 3. Using results of M. Slater [6], this is true without the restriction of characteristic not 3. So when D is alternative $(\text{Rad } D)_n$ is an ideal of D_n and is certainly nilpotent since $(\text{Rad } D)^2$ is an ideal of D properly contained in Rad D.

Since $J\cong H(D_n)$ we have that $J_{ii}\cong H(D)$. If $H(D)/(H(D)\cap \text{Rad } D)$ contains nilpotent elements it must also contain absolute zero divisors, i.e. there is an element $b\in H(D)$ so that $aU_b\in H(D)\cap \text{Rad } D$ for all $a\in H(D)$. Suppose such an element $b\in H(D)$ exists, and let $x=h+k$ be an arbitrary element of D where $h\in H(D)$ and $\bar{k}=-k$. Then $(xb)^3=(h+k)bhb(h+k)b+(h+k)bkbhb+(h+k)b(kbk)b$ is in Rad D since bhb and $b(kbk)b$ are. Therefore xb is nilpotent for all x in D, and thus Db is a nil left ideal of D since b is in the nucleus of D. Hence b must be in Rad D and so finally $H(D)/(H(D)\cap \text{Rad } D)$ contains no absolute zero divisors. So the isomorph of $N_i=\{x\in J_{ii} \mid x \text{ is nilpotent}\}$ is contained in $H(D)\cap \text{Rad } D$. Therefore the isomorph of the ideal N of Theorem 4.2 is contained in

$$H(D_n\cap(\text{Rad } D)_n).$$

Hence N is nilpotent. Now $R=\text{Rad } J$ is A_0+N. But R is nil and $\bar{A}_0+A_0+N/N$ is finite dimensional, hence $\bar{A}_0$ is nilpotent and therefore R is also.

If J is not isomorphic to $H(D_n)$ but to $H(D_n,\gamma)$, $\gamma\neq 1$, then we know that the isomorph $\mathscr{I}$ of R in $H(D_n,\gamma)$ is an ideal not only of $H(D_n,\gamma)$ but also of $H(D_n)$. In fact it is precisely the radical of $H(D_n)$. So the subalgebra of the enveloping algebra of $H(D_n)$ generated by $\mathscr{I}$ is nilpotent. But this subalgebra is identical with the subalgebra generated by $\mathscr{I}$ in the enveloping algebra of $H(D_n,\gamma)$. Therefore $\mathscr{I}$ is nilpotent in $H(D_n,\gamma)$ and hence R is nilpotent.

THEOREM 5.3. *Suppose N is an ideal of J, such that $N^3=0$. Then N has finite dimension.*

Proof. If $N^2=0$ then every element of N is an absolute zero divisor and so every subspace of N is a quadratic ideal and so N has finite dimension.

If $N^2\neq 0$, we consider the ideal N^2+N^2J of J which is contained in N. $N^3=0$ implies that every element of N^2 is an absolute zero divisor and so N^2 has finite dimension since every subspace of N^2 is a quadratic ideal of J.

Now let $a, x \in J$ and let $n \in N^2$. Using the Jordan identity we have $0=2(n, a, na)+(a, a, n^2)=2(na)^2$ since $na\cdot a \in N$. Hence $(na)^2=0$. Linearizing, we see that $na\cdot nb=0$ for any $b \in J$. Also $0=(x, na, na)+(a, na, nx)+(n, na, ax)=(x\cdot na)na+(na\cdot a)nx$. Again $0=(n, x, na\cdot a)+(a, x, na\cdot n)+(na, x, na)=(na\cdot a)nx$. Hence $(x\cdot na)na=0$, so we finally have $xU_{na}=2(x\cdot na)na-x(na)^2=0$ for all $n \in N^2$ and for all $x, a \in J$. Let $n_1, n_2, \ldots, n_k$ be a basis for N^2 over F. Then $N^2J=n_1J+n_2J+\cdots+n_kJ$. Every element in n_iJ is an absolute zero divisor, and so every subspace of n_iJ is a quadratic ideal. Hence n_iJ is finite dimensional, $i=1, 2, \ldots, k$, and so N^2+N^2J has finite dimension.

If $N=N^2+N^2J$ we are done. If $N\neq N^2+N^2J$ then under the natural homomorphism $J\to J/(N^2+N^2J)$, N goes onto $\bar{N}=N/(N^2+N^2J)$. But then $\bar{N}^2=\bar{0}$. Hence again $\bar{N}=N+N^2+N^2J$ has finite dimension and thus N has finite dimension.

THEOREM 5.4. *If N is a nilpotent ideal of J, then N has finite dimension.*

Proof. For any ideal N we construct the descending chain of ideals $M_1\supset M_2\supset\cdots$ by defining $M_1=N$, and $M_{k+1}=M_k^3$, $k=1, 2, \ldots$. Since in our case N is nilpotent, this chain properly descends and has finite length, say t. If $t=2$, i.e. $M_2=N^3=0$, we are done by the preceding theorem. Assume that the theorem is true for all nilpotent ideals with chains of length less than t. In our chain $M_{t-1}^3=M_t=0$ hence M_{t-1} has finite dimension. Under the natural homomorphism $J\to J/M_{t-1}$, N goes to $\bar{N}=N/M_{t-1}$. But the chain for $\bar{N}$ in J/M_{t-1} has length less than t, so by assumption $\bar{N}=N+M_{t-1}$ has finite dimension. Therefore N has finite dimension and we have our theorem.

COROLLARY 5.5. *If $1=\sum_{i=1}^n e_i$ is the identity element of J and the e_i are connected with $n\geqq 3$, then $R=\text{Rad}\, J$ has finite dimension.*

THEOREM 5.6. *Let $J=J_{11}+J_{12}+J_{22}$ be the Peirce decomposition of a Jordan algebra J relative to the orthogonal idempotents e_1 and e_2. Suppose that $J_{12}^2\subset N_1+N_2$ where N_i is a nilpotent ideal of J_{ii}, $i=1, 2$. Then $K=N_1+J_{12}+N_2$ is a nilpotent ideal of J.*

Proof. Clearly K is a subalgebra of J. Also $B=N_1+N_2$ is a finite-dimensional subalgebra of J. Therefore if B^* denotes the subalgebra of the multiplication algebra generated by the right multiplications by elements of B, B^* is nilpotent (see [5, p. 95]). If $J_{12}=BJ_{12}$, then $J_{12}=B(B(\cdots(BJ_{12})\cdots))=(B^*)^kJ_{12}$ for each k and so $J_{12}=0$ and in this case K is a nilpotent ideal of J.

If BJ_{12} is strictly contained in J_{12}, we see that $K^2\subset B+BJ_{12}\subsetneq K$. Define $K^{(1)}=K_{(1)}=K$, and $K^{(j+1)}=[K^{(j)}]^2$, $K_{(j+1)}=[K_{(j)}]^3$. Then clearly $K_{(j)}\subset K^{(j)}$ and $K^{(j)}\subset B+(B^*)^{(j-1)}J_{12}\cdot K$. K is a solvable ideal and there is an ideal $I\neq 0$ of J contained in K so that $I^3=0$. Therefore by an argument similar to that used in the proof of Theorem 5.4 we see that K is finite dimensional and hence nilpotent.

COROLLARY 5.7. *Let $1=\sum_{i=1}^n e_i$ with the e_i primitive orthogonal idempotents, and let $P_1, P_2, \ldots, P_k$ be the partition of $\{e_1, \ldots, e_n\}$ determined by the relation "connected." Then if $|P_i|\geqq 3$, $i=1, \ldots, k$, the radical of J is nilpotent.*

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